1. The curve with equation $y = f(x)$ where

$$f(x) = x^2 + \ln(2x^2 - 4x + 5)$$

has a single turning point at $x = \alpha$

(a) Show that α is a solution of the equation

$$2x^3 - 4x^2 + 7x - 2 = 0$$

(4)

The iterative formula

$$x_{n+1} = \frac{1}{7}(2 + 4x_n{}^2 - 2x_n{}^3)$$

is used to find an approximate value for α.

Starting with $x_1 = 0.3$

(b) calculate, giving each answer to 4 decimal places,

    (i) the value of x_2

    (ii) the value of x_4

(3)

Using a suitable interval and a suitable function that should be stated,

(c) show that α is 0.341 to 3 decimal places.

(2)

a) $f(x) = x^2 + \ln(2x^2 - 4x + 5)$

     $f'(x) = 2x + \dfrac{4x - 4}{2x^2 - 4x + 5}$ ②

turning point has $f'(x) = 0$

$2x + \dfrac{4x - 4}{2x^2 - 4x + 5} = 0$ $\Big)\; x(2x^2 - 4x + 5)$

$2x(2x^2 - 4x + 5) + 4x - 4 = 0$ ①

$4x^3 - 8x^2 + 14x - 4 = 0$

$2x^3 - 4x^2 + 7x - 2 = 0$ ①

b) $x_{n+1} = \frac{1}{7}\left(2 + 4x_n^2 - 2x_n^3\right)$

$x_1 = 0.3$

$x_2 = \frac{1}{7}\left(2 + 4(0.3)^2 - 2(0.3)^3\right)$ ①

(i) $x_2 = 0.329428...$
$= 0.3294$ (4dp) ①

(ii) $x_4 = 0.339823...$
$= 0.3398$ (4dp) ①

c) let $h(x) = 2x^3 - 4x^2 + 7x - 2$
$h(x) = 0$ represents a turning point of $f(x)$.
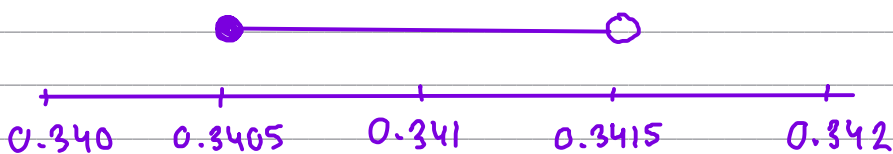want to show that $\alpha = 0.341$ to 3dp.

$h(0.3415) = 0.00366...$ > 0
$h(0.3405) = -0.00130...$ < 0 ①

- since there is a change in sign
- and $f'(x)$ is a continuous function
- $\alpha = 0.341$ to 3dp ①

0.340   0.3405   0.341   0.3415   0.342

2. A curve has equation $y = f(x)$, where

$$f(x) = \frac{7xe^x}{\sqrt{e^{3x} - 2}} \qquad x > \ln\sqrt[3]{2}$$

(a) Show that

$$f'(x) = \frac{7e^x(e^{3x}(2 - x) + Ax + B)}{2(e^{3x} - 2)^{\frac{3}{2}}}$$

where A and B are constants to be found.

(5)

(b) Hence show that the x coordinates of the turning points of the curve are solutions of the equation

$$x = \frac{2e^{3x} - 4}{e^{3x} + 4}$$

(2)

The equation $x = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ has two positive roots α and β where $\beta > \alpha$

A student uses the iteration formula

$$x_{n+1} = \frac{2e^{3x_n} - 4}{e^{3x_n} + 4}$$

in an attempt to find approximations for α and β

Diagram 1 shows a plot of part of the curve with equation $y = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ and part of the line with equation $y = x$

(c) draw a staircase diagram to show that the iteration formula starting with $x_1 = 1$ can be used to find an approximation for β

(1)

Use the iteration formula with $x_1 = 1$, to find, to 3 decimal places,

(d) (i) the value of x_2

(ii) the value of β

(3)

Using a suitable interval and a suitable function that should be stated

(e) show that $\alpha = 0.432$ to 3 decimal places.

(2)

a) $f(x) = \dfrac{7xe^x}{(e^{3x}-2)^{1/2}}$

$\dfrac{d}{dx}(7xe^x):$ let $u = 7x$ $v = e^x$

$\qquad\qquad\qquad \dfrac{du}{dx} = 7 \qquad \dfrac{dv}{dx} = e^x$

$\qquad\qquad\qquad\qquad\quad ①\qquad\qquad ①$

$\dfrac{du}{dx}v + \dfrac{dv}{dx}u = 7e^x + 7xe^x$

$\dfrac{d}{dx}\left((e^{3x}-2)^{1/2}\right) = \dfrac{1}{2} \times 3e^{3x} \times (e^{3x}-2)^{-1/2}$ ①

$\qquad\qquad\qquad\qquad = \dfrac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}$

$f(x) = \dfrac{7xe^x}{(e^{3x}-2)^{1/2}}$ using quotient rule   ①

$f'(x) = \dfrac{(e^{3x}-2)^{1/2}(7e^x+7xe^x) - 7xe^x\left(\frac{3}{2}e^{3x}(e^{3x}-2)^{-1/2}\right)}{e^{3x}-2}$

$= \dfrac{7(e^{3x}-2)^{-1/2}\left[e^x(e^{3x}-2)(1+x) - \frac{3}{2}xe^xe^{3x}\right]}{e^{3x}-2}$

                                                          factoring out
                                                         $7(e^{3x}-2)^{-1/2}$

$= \dfrac{7e^x\left[(e^{3x}-2)(1+x) - \frac{3}{2}xe^{3x}\right]}{(e^{3x}-2)^{3/2}}$ *factoring out e^x*

moving $(e^{3x}-2)^{-1/2}$ to the denominator

Only use the copy of Diagram 1 if you need to redraw your answer to part (c).
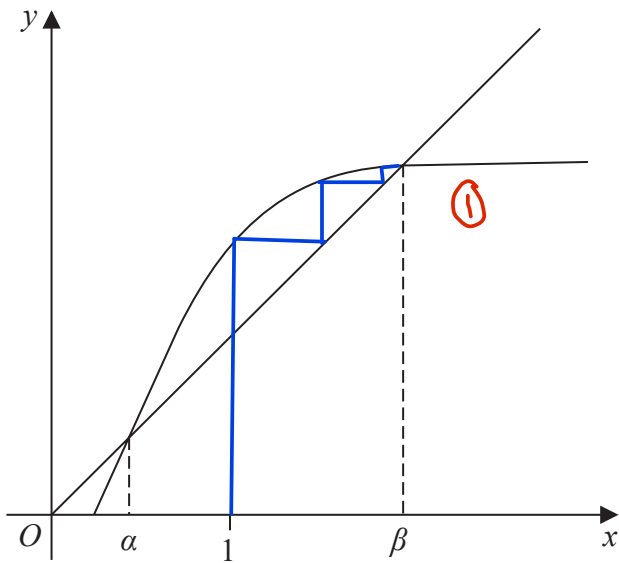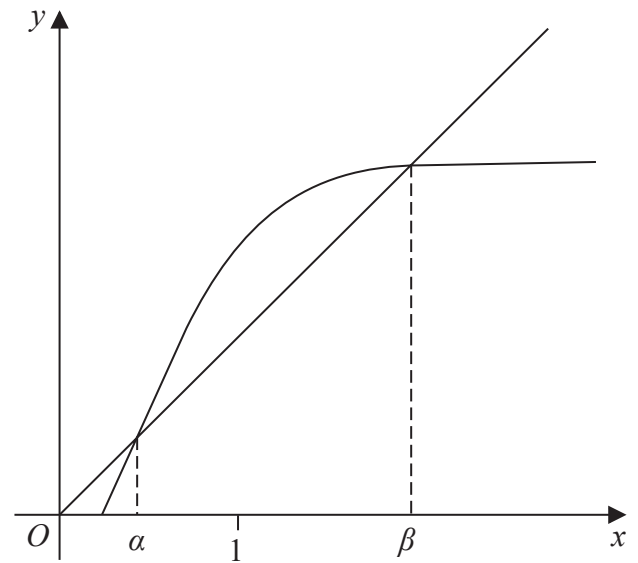
Diagram 1

copy of Diagram 1

$$f'(x) = \frac{7e^x\left[e^{3x} + xe^{3x} - 2 - 2x - \frac{3}{2}xe^{3x}\right]}{\left(e^{3x} - 2\right)^{3/2}}$$

expanding bracket

$$= \frac{7e^x\left[e^{3x} - \frac{1}{2}xe^{3x} - 2x - 2\right]}{\left(e^{3x} - 2\right)^{3/2}}$$

collecting like terms

$$= \frac{7e^x\left(2e^{3x} - xe^{3x} - 4x - 4\right)}{2\left(e^{3x} - 2\right)^{3/2}}$$

multiplying top and bottom by 2

$$= \frac{7e^x\left(e^{3x}(2 - x) - 4x - 4\right)}{2\left(e^{3x} - 2\right)^{3/2}}$$

as required.

b) turning points have $f'(x) = 0$

$\Rightarrow \dfrac{7e^{x}\left(e^{3x}(2-x) - 4x - 4\right)}{2(e^{3x}-2)^{3/2}} = 0$

$e^{x} \neq 0$,
multiply by
$2\left(e^{3x}-2\right)^{3/2}$

$e^{3x}(2-x) - 4x - 4 = 0$

$2e^{3x} - xe^{3x} - 4x - 4 = 0$

$x(e^{3x} + 4) = 2e^{3x} - 4$ ①

$x = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$ ①

c) drawn on diagram

d) $x_{n+1} = \dfrac{2e^{3x_n} - 4}{e^{3x_n} + 4}$

(i) $x_2 = \dfrac{2e^{3} - 4}{e^{3} + 4} = 1.50177\ldots$ ①

$= 1.502 \ (3dp)$ ①

(ii) $\beta = 1.96757\ldots$
$= 1.968 \ (3dp)$ ①

e) α is a solution of $x = \dfrac{2e^{3x} - 4}{e^{3x} + 4}$

$\therefore$ a solution of $\dfrac{2e^{3x} - 4}{e^{3x} + 4} - x = 0$

so define $h(x) = \dfrac{2e^{3x}-4}{e^{3x}+4} - x$, so $h(\alpha) = 0$.

$h(0.4315) = -0.000297... < 0$
$h(0.4325) = 0.000947 > 0$ (1)

- since there is a change of sign
- and $h(x)$ is continuous
- $\alpha = 0.432$ (to 3dp)   (1)